SURGERY ON LINKS AND DOUBLE BRANCHED COVERS OF S^3

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§0. *Introduction*

This paper deals with the relationship between 2-fold cyclic coverings of S^3 branched over a link and closed, orientable 3-manifolds which are obtained by doing surgery on a link in S^3. In Theorem 1 it is shown that every 2-fold cyclic branched covering of S^3 can be obtained by doing surgery on a "strongly invertible" link, that is, a link L which has the property that there is an orientation preserving involution of S^3 which induces in each component of L an involution with two fixed points. This result has some interesting consequences. Let K be a non-trivial knot in S^3. Then Theorem 1, which is a constructive result, allows us to obtain a link L in S^3 such that the 2-fold covering space $\tilde{K}$ of S^3 branched over K can be obtained by doing surgery on L. Note that if L has property P, then $\tilde{K}$ cannot be a counterexample to Poincaré Conjecture because $\pi(\tilde{K}) \neq 1$. Thus, every simply connected 2-fold cyclic covering of S^3 is S^3 iff every strongly invertible link has property P (Corollary 1). As a second consequence of Theorem 1 we obtain a new proof of a result established earlier by Viro [25] and also by Birman and Hilden [2], that every closed, orientable 3-manifold of Heegaard genus ≤ 2 is a 2-fold cyclic branched covering of S^3 (Corollary 2). In Corollary 3 we will sharpen Theorem 1 showing that every 2-fold cyclic branched covering of S^3 can be obtained by doing surgery on a member of a special family of strongly invertible links in S^3.

Let L be a link such that there is an orientation preserving involution of S^3 with fixed points which induces an involution in each component of L. Let M be a manifold that is obtained by doing surgery

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on L. We will see in Theorem 2 that M is a 2-fold cyclic covering of a manifold that is obtained by doing surgery on a link in S^3. As an application of Theorem 2, it is shown that each manifold that is obtained by doing surgery on a noninvertible pretzel knot or on the noninvertible "borromeans rings" is a 2-fold cyclic branched covering of a 2-fold cyclic branched covering of S^3. This yields some insight into the answer to a question (Question 3) raised by Birman and Hilden.

The construction of the link L in Theorem 1 uses some knot modifications, defined by Wendt, which have the effect of changing K into the trivial knot. Having in mind the purpose of finding, for a given knot K, if $\pi(\tilde{K})$ is or is not trivial, we define in Section 3 some modifications of a knot which generalize Wendt's modifications. These modifications have the effect of exhibiting $\tilde{K}$ as a manifold which is obtained by doing "generalized surgery" on a link in S^3, that is, removing n disjoint solid tori from S^3 and replacing each torus with a special "graph-manifold" which is bounded by a torus. The advantage of this is that if a link has property P, then a counterexample to the Poincaré conjecture cannot be obtained by doing generalized surgery on it (Theorem 4).

This fact allows us, in Section 4, to establish that there cannot be a counterexample to the Poincaré Conjecture among the 2-fold cyclic coverings of S^3 which are branched over the knots of Kinoshita-Terasaka (Section 4.1), or over Conway's 11-crossing knot with Alexander polynomial 1 (see Section 4.2), or over a special class of closed 3-braids (see Section 4.3) first studied by Birman and Hilden.

In Section 5 it is established that graph-manifolds are in the Poincaré Category. This fact was used earlier in the paper, in the proof of Theorem 4.

Acknowledgement. I would like to express my deep gratitude to Professor J. S. Birman for her valuable suggestions and comments in reading my manuscript.

§1. *Statement of the problems*

In this section we will discuss several interesting questions which have been posed by Ralph Fox and others about the Poincaré Conjecture and related matters. These questions will serve to motivate the main results of this paper, which are given in Sections 2, 3, 4 and 5, below.

Let L denote a link in S^3, and let $\tilde{L}$ denote the 2-fold cyclic covering space of S^3 branched over L. Since 2-fold branched covering spaces are in many ways especially simple (see [2, 5, 6, 15, 25, 27]), one might like to know how they are related to the class of *all* closed, orientable 3-manifolds? Ralph Fox has proved [6] that the 3-dimensional torus $S^1 \times S^1 \times S^1$ is not a 2-fold cyclic branched cover of S^3. However he has given a conjecture [6, Conjecture A′] that implies an affirmative answer to the question:

Question 1. Is every closed, orientable, simply-connected 3-manifold a 2-fold cyclic branched cover of S^3?

This appears to be a deep and difficult question, and, as will be seen below, it may even be equivalent to the Poincaré Conjecture.

Now, in [17], [18] it was shown that there are Seifert fiber spaces, different from $S^1 \times S^1 \times S^1$, which are not 2-fold cyclic coverings of S^3. However, all of them, are 2-fold cyclic coverings branched over a 3-sphere with handles [18].

Question 2. Is every closed, orientable 3-manifold a 2-fold cyclic covering branched over a 3-sphere with handles?

If Question 2 has an affirmative answer, then *each closed, orientable 3-manifold* M *with* $H_1(M)$ *finite is a 2-fold cyclic covering of* S^3, because the lift to M of a non-separating 2-sphere (in S^3 with $g > 0$ handles) must be a non-separating closed, orientable surface in M. Thus $H_2(M)$ and $H_1(M)$ are infinite. Then, an affirmative answer to Question 2 implies an affirmative answer to Question 1.

Note that a 3-sphere with $g > 0$ handles is a 2-fold cyclic branched covering of S^3. Joan S. Birman and Hugh M. Hilden have suggested that it is reasonable to ask the following question, which looks like a weaker question than Question 2.

Question 3. Is every closed, orientable 3-manifold a 2-fold branched cyclic covering of a 2-fold branched cyclic covering of $\cdots$ of a 2-fold branched cyclic covering of S^3?

It was observed by Birman and Hilden that if the answer to Question 3 is affirmative, then Fox's argument [5] implies that *if a counterexample exist to the Poincaré Conjecture, then there is also a counterexample which is a 2-fold branched cyclic covering of* S^3.

Thus an affirmative answer to one of the three above questions would reduce the investigation of the Poincaré Conjecture, to the case of 2-fold cyclic coverings of S^3.

Now, the trivial knot is the only knot which has S^3 as associated 2-fold cyclic covering branched over it [27]. On the other hand, if L has more than one component, then $H_1(\tilde{L}) \neq 0$ [6] and if $L = L_1 \# L_2$ is a composite knot, then $\pi(\tilde{L}) = \pi(\tilde{L}_1) * \pi(\tilde{L}_2)$ [15, Theorem V.5.3.]. Thus, one is led to consider the following Conjecture (see [15, Conjecture I.1.1.]):

CONJECTURE 1. If N is a non-trivial prime knot, then $\pi(\tilde{N}) \neq 1$.

If one searches for a counterexample to Questions 2, 3, then one need not consider Seifert fiber spaces or closed *graph-manifolds* ("Graphen-mannigfaltigkeiten," see [26]) because all of them are 2-fold cyclic coverings of S^3 with handles.[1] I suggest looking for M among the closed, orientable 3-manifolds obtained by doing surgery on a knot in S^3.

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1 In [18] this was proved for Seifert manifolds and for graph-manifolds M represented by a graph $A(M)$. Of course, this can be extended to each closed graph-manifold according to [26, Satz 6.3, p. 88] and [15, Teorema V.5.3.] and [25; 3.10].

Therefore, in this paper, we explore the relationship between 2-fold cyclic coverings of S^3 branched over a link and closed, orientable 3-manifolds which are obtained by doing surgery on a link in S^3.

§2. *Surgery on links and double branched covers of* S^3

Let L be a link in S^3. L is called *strongly-invertible* if there is an orientation-preserving involution of S^3 which induces in each component of L an involution with two fixed points. Every strongly-invertible link L is invertible, but I do not know if every invertible link is a strongly-invertible link.

THEOREM 1. *Let* M *be a closed, orientable 3-manifold that is obtained by doing surgery on a strongly-invertible link* L *of* n *components. Then* M *is a 2-fold cyclic covering of* S^3 *branched over a link of at most* $n+1$ *components. Conversely, every 2-fold cyclic branched covering of* S^3 *can be obtained in this fashion.*

Proof of Theorem 1. Let S^3 be represented as Euclidean space with an ideal point at infinity. It can be supposed without loss of generality [27], that there is an axis E in S^3 such that the axial symmetry u with respect to E induces in each component of L an involution with two fixed points. For the sake of brevity, the first part of Theorem 1 will be proved for a knot N in S^3.

Let U(N) be a regular neighborhood of N such that u induces an involution in U(N) (a typical case is illustrated in Figure 1a). Let V be the solid torus, as represented in Figure 1b, and let u' be the symmetry with respect to the axis E'. There is a homeomorphism ψ of $\partial U(N)$ onto ∂V such that $(u'|\partial V)\psi = \psi(u|\partial U(N))$.

Let ϕ now be a homeomorphism of ∂V onto $\partial U(N)$. Then $\psi\phi$ is an autohomeomorphism of ∂V and it can be supposed (by composing ϕ, if necessary, with an isotopy) that

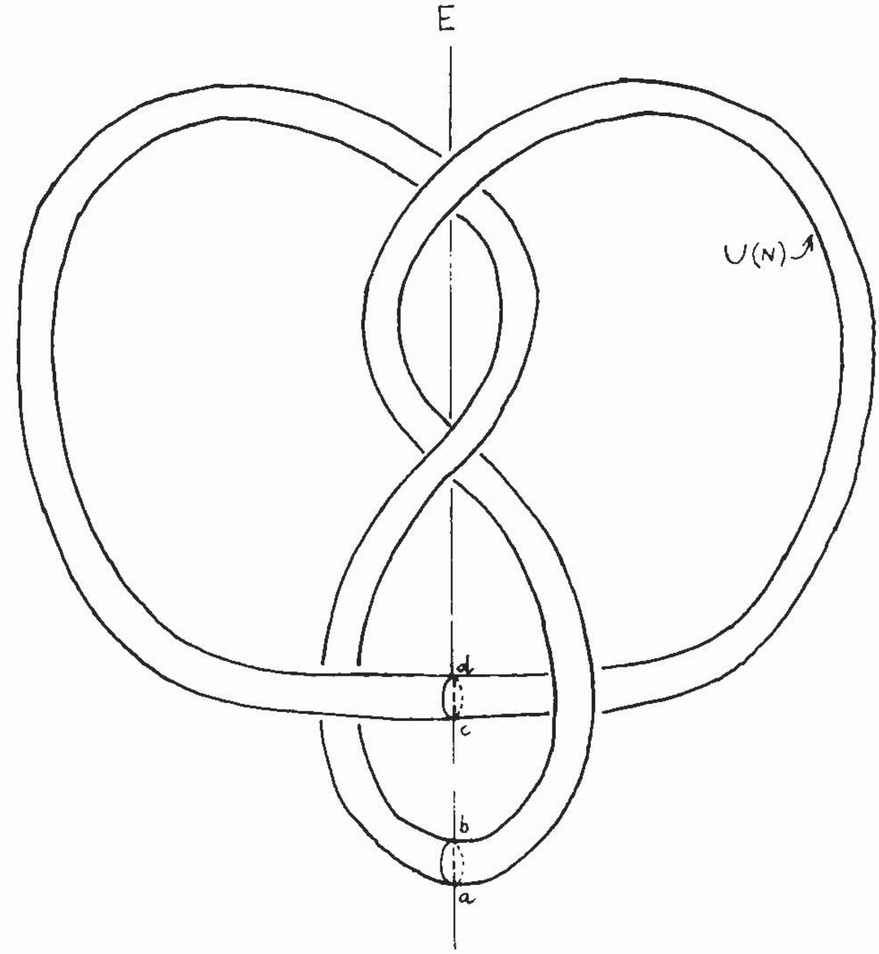

Fig. 1a.

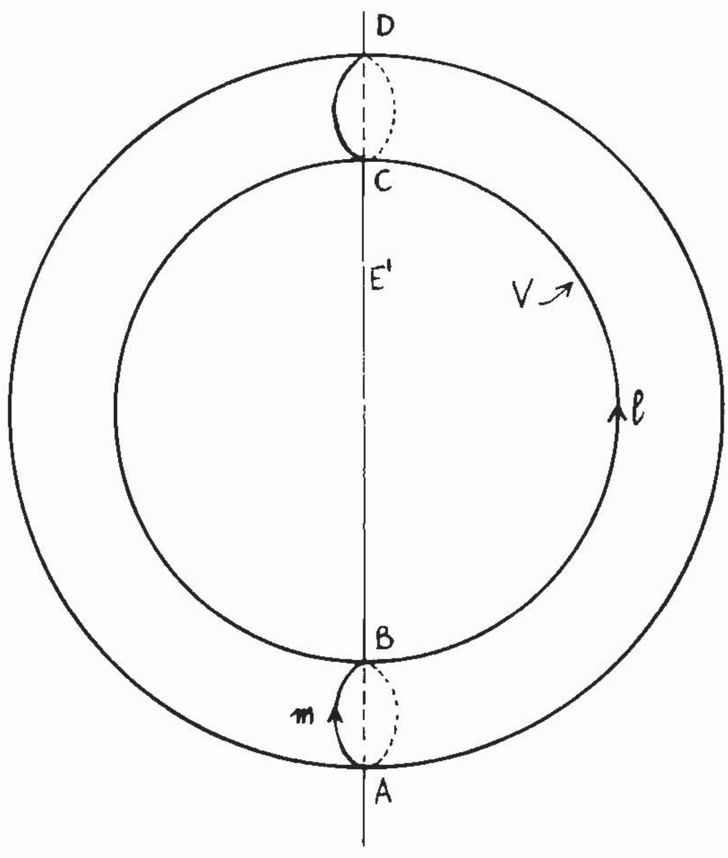

Fig. 1b.

$$(\psi\phi)(u'|\partial V) = (u'|\partial V)(\psi\phi)\,.$$ [2]

Thus $\phi(u'|\partial V) = \psi^{-1}(u'|\partial V)\psi\phi = (u|\partial U(N))\phi$.

Then, the space M obtained by pasting V to $S^3 - U(N)$ by means of ϕ is compatible with the involutions u and u′, and admits an involution u″, induced by u and u′. The orbit-space of $(S^3 - U(N)) \cup V$ under u″ can be obtained by adjoining the orbit space of V under u′ (which is a ball) to the orbit-space of $S^3 - U(N)$ under u, which is S^3 minus a ball (see in Figure 1c a fundamental set for the action of u on U(N)). Then M is a 2-fold cyclic covering of S^3, branched over the image of $E - (ab + cd) + (AB + CD)$ (see Figures 1a and 1b). This is a link in S^3 which has, at most, two components.

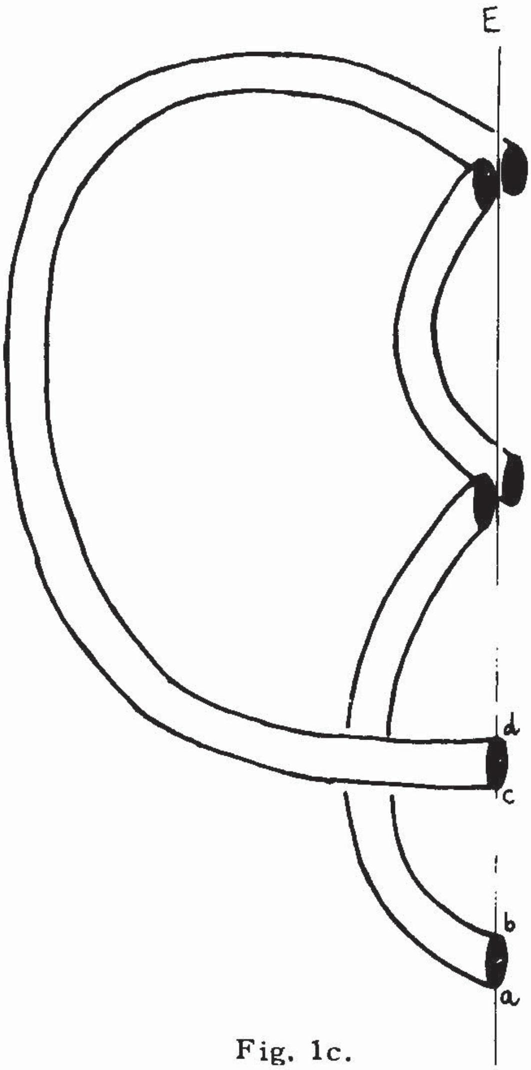

Fig. 1c.

Conversely, suppose that M is a 2-fold cyclic covering of S^3, branched over a link L. We consider two ways to modify this link, by removing certain solid balls from S^3 and sewing them back differently. First, it is possible, by applying modifications of type W_1 (see Figure 2a), to change a given link L in S^3 into a knot K in S^3. Then, by

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2 This result is contained implicitly in [3], and is proved in [2], [25] and [18]. In [2] and [25] this result has been generalized for orientable surfaces of genus $g \leq 2$. For $g > 2$ this generalization is not true in general (see [6] and [17]).

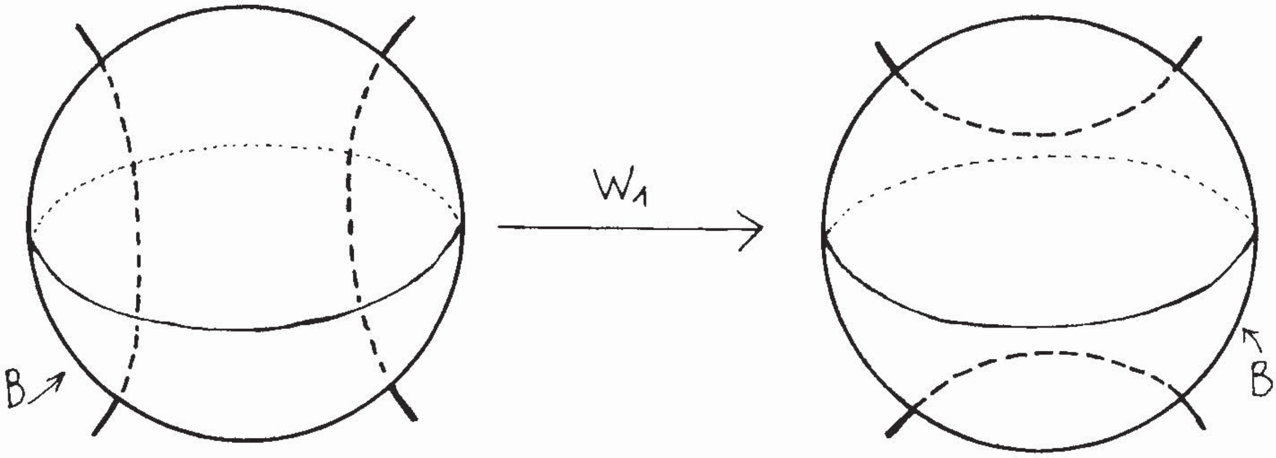


Fig. 2a.

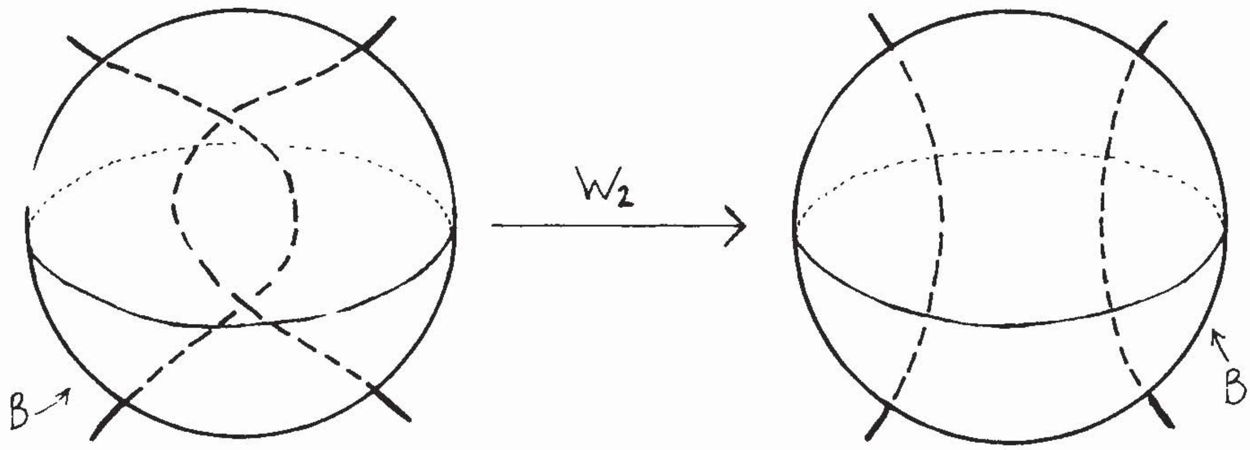


Fig. 2b.

applying further modifications of type W_2 (see Figure 2b), it is possible to change the knot K into the trivial knot T (see [28]). Let n be the minimal number of modifications of type W_1, W_2 that are necessary in order to change the given link L into the trivial knot T.

It may be supposed that these modifications are set up in the inner of n disjointed balls $B_1, \cdots, B_n$ of S^3 (see Figure 2). Note that the 2-fold cyclic coverings of B_i branched over $B_i \cap T$ are solid tori. Thus, in order to build up $\tilde{L}$ it is sufficient to do surgery along n solid tori in $\tilde{T} = S^3$.

Let $\tilde{B}_i$ be the 2-fold cyclic covering of B_i, branched over $B_i \cap T$. Then $\bigcup_{i=1}^{n} \tilde{B}_i$ can be interpreted as a regular neighborhood of a strongly-invertible link in S^3. Thus, $\tilde{L}$ can be obtained by doing surgery on a strongly-invertible link in S^3 which has, at most, n components. □

Recall that a link L in S^3 has *property* P when it is not possible to obtain a counterexample to the Poincaré Conjecture by doing surgery on it.

COROLLARY 1. *Conjecture 1 is true iff every strongly invertible link has property* P. □

As property P is known to be true for many links [1], [8], [23], Corollary 1 implies that Conjecture 1 can be established for a large family of knots. In Section 4 we will apply Theorem 1 in this way to establish that there cannot be a counterexample to the Poincaré Conjecture among the 2-fold coverings of S^3 which are branched over the knots of Kinoshita-Terasaka (see Section 4.1), or over Conway's 11-crossing knot with Alexander polynomial 1 (see Section 4.2), or over a special class of closed 3-braids (see Section 4.3).

We now give a different application of Theorem 1. Let $g \geq 1$ be an integer. Let L be a link in $R^3 = S^3 -$ (one point) made up of a disjoint union of circles, each being one of the following: (i) a circle of radius < 1, center at $(2n+1, 0, 0)$ where $0 \leq n \leq g$, and lying in the x, z plane, or (ii) a circle of radius < 1, center at $(2n, 0, 0)$ where $1 \leq n \leq g$, and lying in the x, y plane, or (iii) a circle of radius ≤ 2, center at $(2n, 2, 0)$ where $1 < n < g$, and lying in a parallel plane P_n to the y, z plane. We assume also that the annulus determinated by two concentric components of L must be cut by some other component in exactly one point. Let $\mathcal{L}_g$ be the family of links defined in this way, for a given g. It was proved by Lickorish [13] that every closed, orientable 3-manifold of genus g may be obtained by doing surgery on a link in the class $\mathcal{L}_g$. Let $\mathcal{L}'_g$ be the subfamily of $\mathcal{L}_g$ consisting of those links whose components in P_n have radius 2. Note that a link in $\mathcal{L}'_g$ is strongly-invertible.

Since $\mathcal{L}'_g = \mathcal{L}_g$ for $g \leq 2$, then according to Theorem 1, we obtain another proof of the following result by Viro [25] and Birman and Hilden [2]:

COROLLARY 2. *Every closed, orientable 3-manifold of genus* ≤ 2 *is a 2-fold cyclic branched covering of* S^3. □

COROLLARY 3. *Each 2-fold cyclic covering branched over* S^3 *can be obtained doing surgery on a link in* $\mathcal{L}'_g$, *for some* $g \geq 1$.

Proof of Corollary 3. First, we recall the definition of a "plat on 2m strings." If we represent S^3 as $R^3 + \infty$, then the x, y plane separates S^3 in two balls B_1 and B_2, B_1 containing the positive part of axis z. Let C be a collection of m circles in the x, z plane of radius 2 and centers at points $(1 + 5i, 0, 0)$, where $0 \leq i \leq m-1$. Let f be any orientation-preserving autohomeomorphism of ∂B_1 which keeps the set $C \cap \partial B_1$ fixed as a set. Since f is isotopic to the identity map in ∂B_1, there is an autohomeomorphism $F' : \partial B_1 \times [0, 1] \to \partial B_1 \times [0, 1]$ such that $F'(x,t) = (x',t)$, $F'(x,1) = (x,1)$ and $F'(x,0) = (fx,0)$. Then F' is extended by the identity map outside $\partial B_1 \times [0,1]$ to an autohomeomorphism F of B_1. The subset $L = F(C \cap B_1) \cup (C \cap B_2)$, which is a link in S^3, is called a *plat on* 2m *strings* (for further details, see [2]). It is a known result (see, for instance, [2]) that every link type is represented by at least one plat. Note that $F(C \cap (\partial B_1 \times [0,1]))$ is a geometric braid on 2m strings Thus a plat on 2m strings can be exhibited as a geometric braid on 2m strings by joining the initial points in pairs, and also the terminal points in pairs.

The proof of Corollary 3 may be illustrated by the following example (the general case is left to the reader). Let us consider the plat on 8 strings of Figure 3a. It is possible to change L into the trivial knot by removing ten solid balls B_i $(i = 1, \cdots, 10)$ from S^3 and sewing them back differently (see Figures 3a, 3b, 4a). Note that the 2-fold covering of B_i branched over $B_i \cap L$ or $B_i \cap T$ are solid tori. It is clear that we can

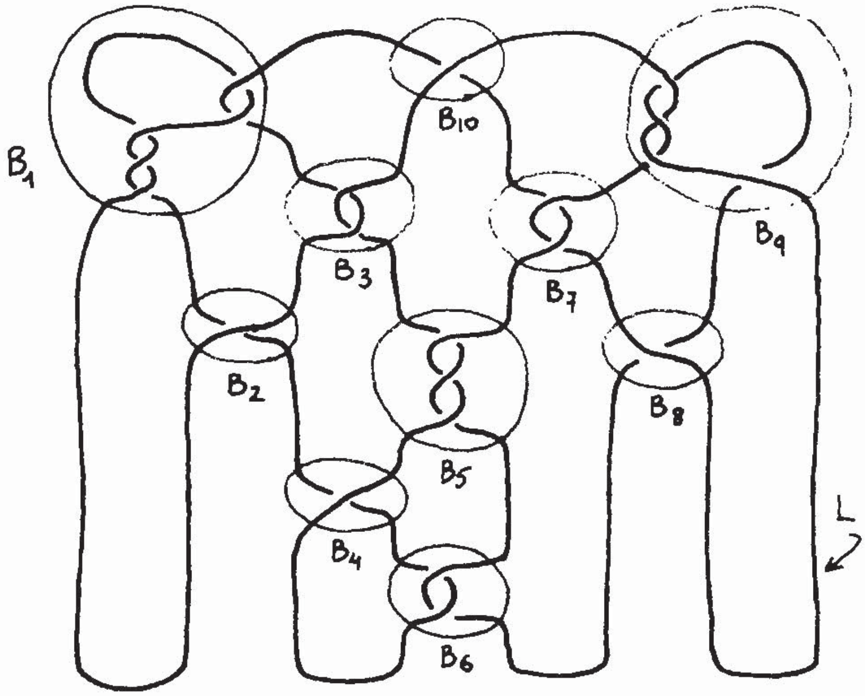


Fig. 3a.

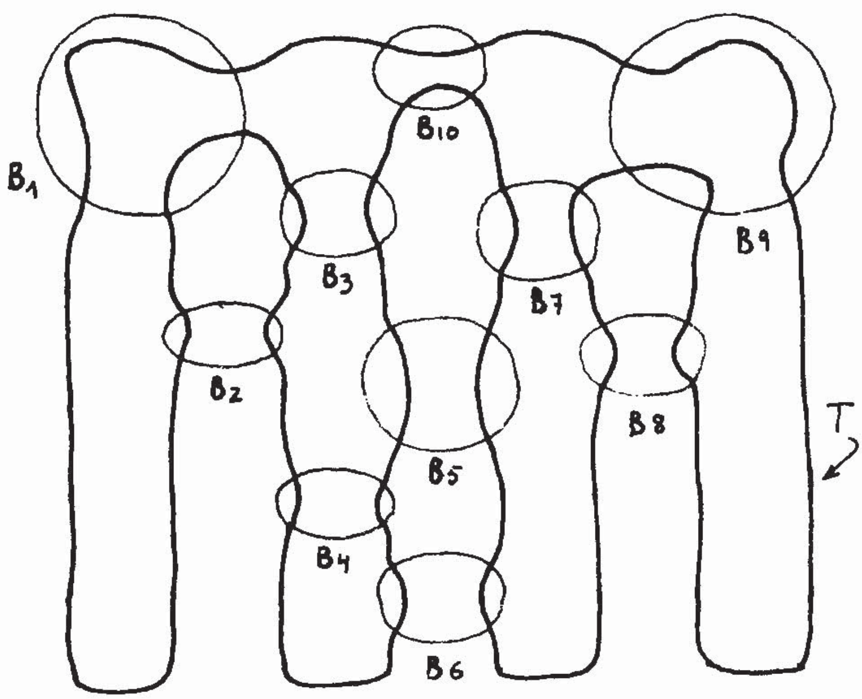
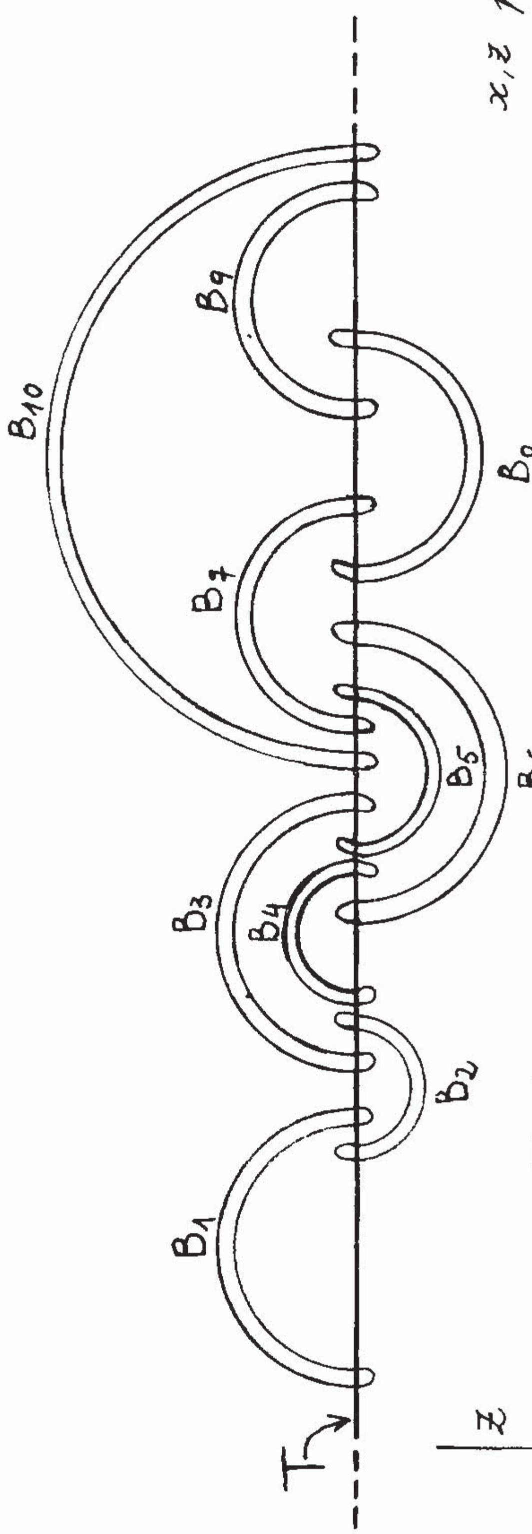
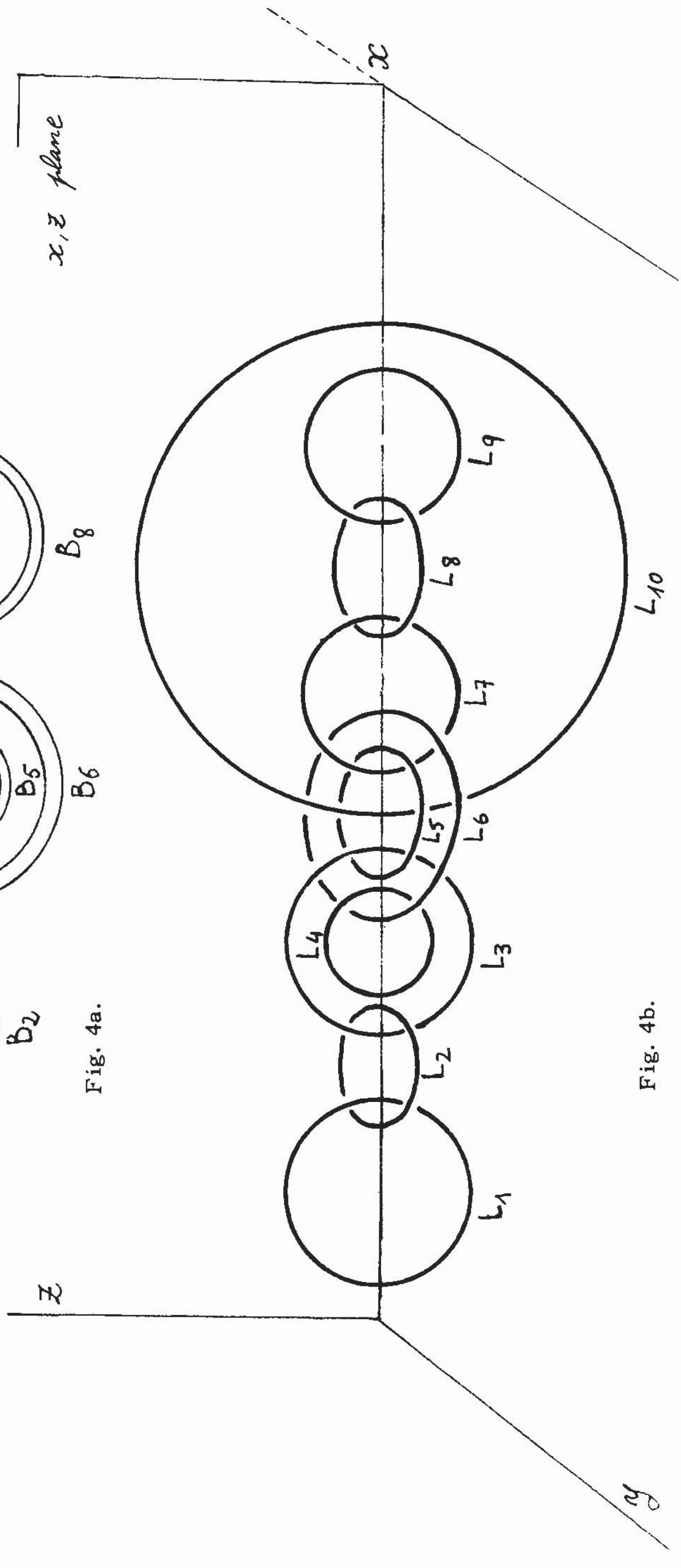


Fig. 3b.



T
B1
B2
B3
B4
B5
B6
B7
B8
B9
B10

Fig. 4a.



x
x, z plane
z
y
L1
L2
L3
L4
L5
L6
L7
L8
L9
L10

Fig. 4b.

obtain $\tilde{L}$ by doing surgery on the link in $\mathfrak{L}'_3$ of Figure 4b. In general, if L is a plat on $2m$ strings then $\tilde{L}$ can be obtained by doing surgery on a link in $\mathfrak{L}'_{m-1}$.[3] □

As a consequence of Corollary 3 we have:

COROLLARY 4. *Conjecture 1 is equivalent to the Conjecture that each member of* $\mathfrak{L}'_g$, $g \geq 1$, *has property* P.

To explore further the implications of Theorem 1, observe that if there is a closed, orientable 3-manifold M which gives a negative answer to Questions 2 or 3, it must be obtained by doing surgery on a link which is not strongly invertible. This suggests that one study Questions 2 or 3 by studying the 3-manifolds obtained by doing surgery on a non-invertible link.

Let L be a link in S^3 and let suppose that there is an orientation-preserving involution u in S^3, with fixed points, which induces an involution in each component of L. Let L' be the link consisting of those components of L for which the number of fixed points of u is different from two. Let $p: S^3 \to S^3$ the 2-fold cyclic branched covering of S^3 defined by u.

THEOREM 2. *Every manifold obtained by doing surgery on a link* L *is a 2-fold cyclic covering branched over a manifold obtained by doing surgery on* $p(L')$.

REMARK. Theorem 1 is a special case of Theorem 2.

[3] J. S. Birman has pointed out to me that it is interesting to note that the class of 3-manifolds which are obtained by doing surgery on links in $\mathfrak{L}'_g$ are exactly the class of 3-manifolds which are "2-symmetric" in the notation of [2].

Proof of Theorem 2. For the sake of brevity, suppose that L has only one component and that either u is without fixed points in L or leaves each point of L fixed. Let U(L) be a regular neighborhood of L, such that u induces in U(L) an involution. Let $u' = u|\partial U(L)$.

Let V be a solid torus (see Figure 5) whose core C is a circle in the x, y plane with center 0 and radius one. Let z (resp. v) be the involution of V induced by the symmetry with respect to axis OZ (resp. C). There is a homeomorphism ψ of $\partial U(L)$ onto ∂V such that $z\psi = \psi u'$. Let $p = \psi^{-1}P$ and $m = \psi^{-1}M$ be a pair of simple oriented curves in $\partial U(L)$ (see Figure 5).

We now paste V to $S^3 - U(L)$ in the way that M is homologous to $\alpha m + \beta p$, where α and β are coprime integers. It is easy to see that there is a homeomorphism ϕ of ∂V onto $\partial U(L)$ such that $\phi(M) \sim \alpha m + \beta p$ and $\phi^{-1}\psi^{-1}z\psi\phi$, that is $\phi^{-1}u'\phi$, is equal to z if α is odd, or is equal to v if α is even.

Let W be the space obtained by pasting $S^3 - U(L)$ to V by ϕ. The map ϕ is compatible with the involutions u and z (or v, as the case may be). Thus, there is an involution u'' of W, the orbit-space of which is obtained by adjoining the orbit-space of u (that is S^3 minus a solid torus) with the orbit-space of z (or v, as the case may be), which is a solid torus. □

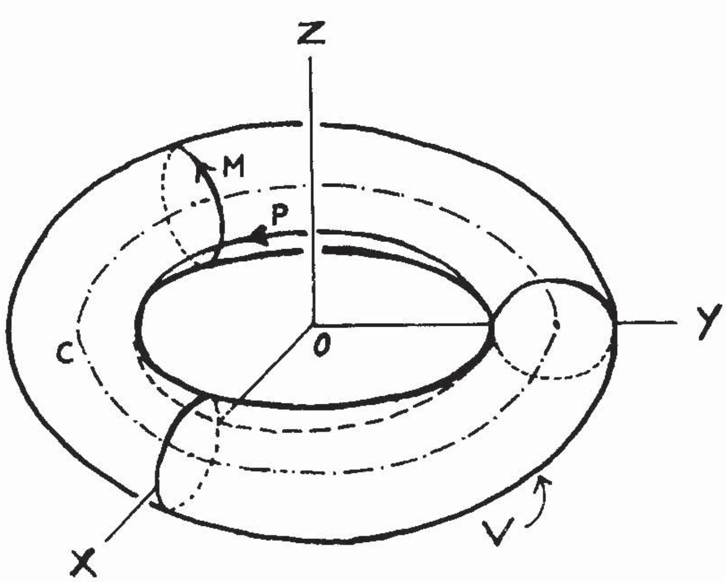


Fig. 5

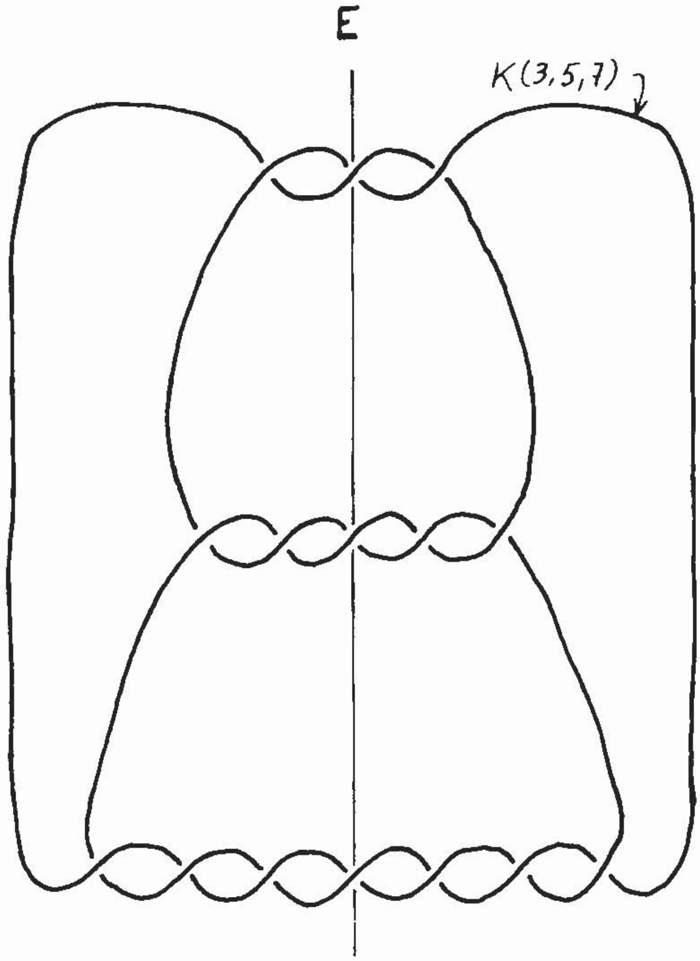


Fig. 6a.

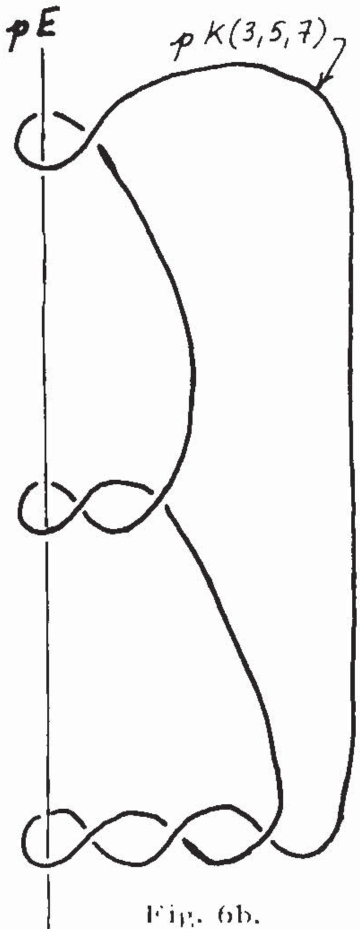


Fig. 6b.

As an application of Theorem 2 consider the pretzel knot $K(p, q, r)$ (see [24]). If any of the numbers p, q, r is even, it is clear that $K(p,q,r)$ is a strongly-invertible link. Thus, one obtains a 2-fold cyclic covering branched over S^3 by doing surgery on $K(p, q, r)$. If the numbers p, q, r are all odd, then there is an involution u of S^3 which induces in the knot $K(p, q, r)$ an involution without fixed points. (A typical case is illustrated in Figure 6a). Thus, every manifold that is obtained by doing surgery on $K(p, q, r)$ is a 2-fold cyclic covering branched over a manifold that is obtained by doing surgery on the trivial knot $p(K(p, q, r))$, where p is the covering defined by u (see Figure 6b). As the trivial knot is strongly-invertible it follows that the manifold obtained by doing surgery on $K(p, q, r)$, $(p, q, r$ odd), is a 2-fold cyclic covering of a 2-fold cyclic

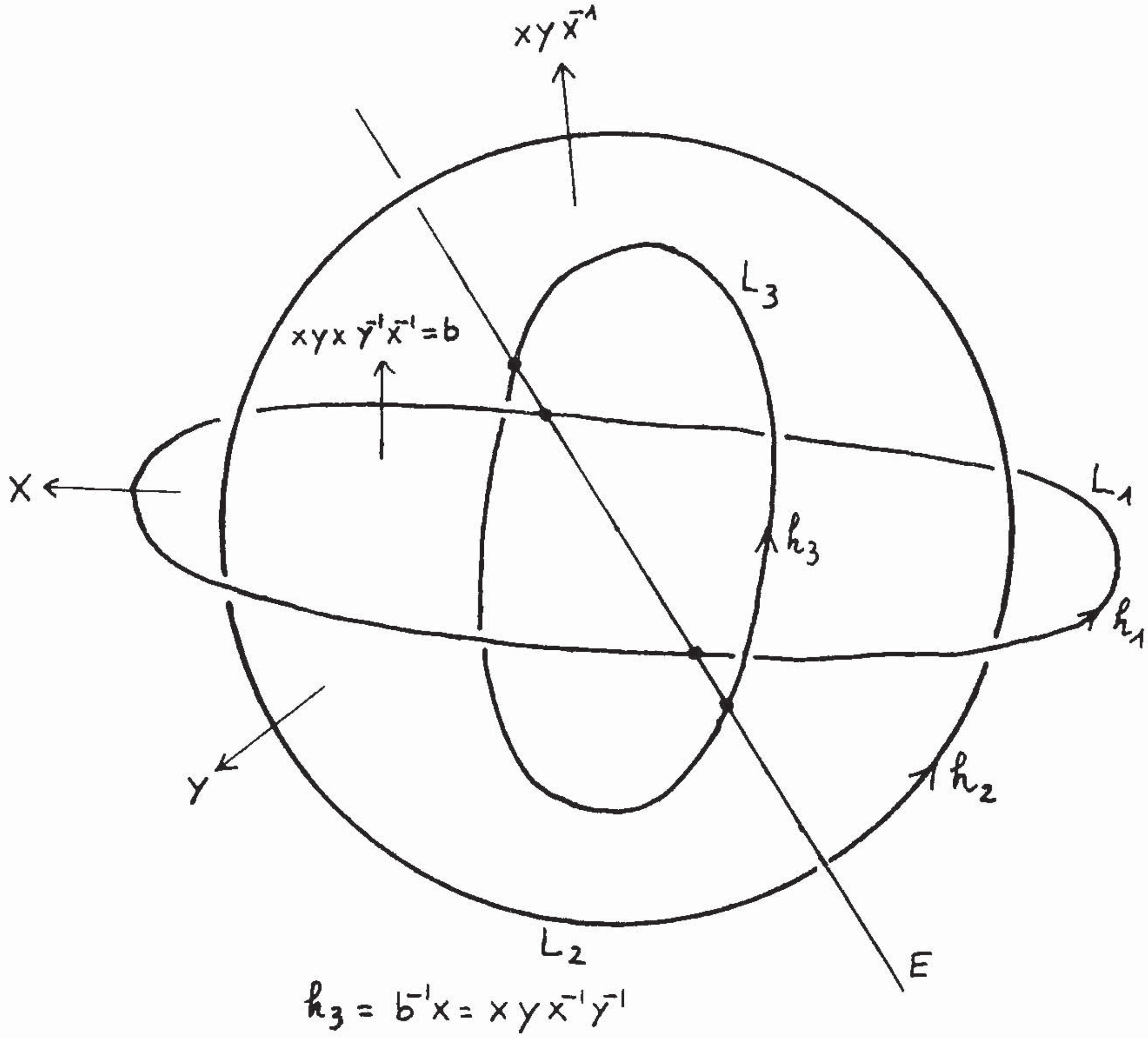


Fig. 7

covering of S^3. This confirms Question 3. Note that Trotter has shown that $K(p,q,r)$ is non-invertible if p, q, r are distinct odd integers, each to greater than one. The author does not know whether the manifolds obtained by doing surgery on these knots are also representable as 2-fold cyclic branched covering of S^3.

As a second application, consider the manifold obtained by doing surgery on the "borromeans rings," B, illustrated in Figure 7. If we remove the solid tori $U(L_1)$, $U(L_2)$, $U(L_3)$ from S^3 and sew them back in such a way that the curves h_1, h_2, h_3 are identified with meridians, then we obtain $S^1 \times S^1 \times S^1$ [12], which is *not* a 2-fold branched cyclic covering of S^3 [6]. This shows that B is not a strongly invertible link. But the axial symmetry with respect to axis E (see Figure 7) induces in each component of B an involution. Then, by Theorem 2, every manifold that is obtained by doing surgery on B is a 2-fold cyclic branched covering of a manifold that is obtained by doing surgery on the trivial knot and this confirms Question 3. For instance, $S^1 \times S^1 \times S^1$ is a 2-fold cyclic branched covering of $S^1 \times S^2$.

It is interesting to note that not only is B non-invertible,[4] but also there is no orientation-preserving involution of S^3 which induces an involution in each component of B and which keeps fixed exactly two points of B.

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[4] To the author's knowledge, this fact has not been established elsewhere in the literature. To prove it, let $F_2 = \{x,y/-\}$ be the group of the link formed by the components L_1, L_2. The group F_2 is a free group on two generators and the element $xyx^{-1}y^{-1}$ is represented by the loop h_3. If ϕ is an automorphism of F_2, then by [14, Theorem 3.9, p. 165], $\phi(xyx^{-1}y^{-1}) = w(xyx^{-1}y^{-1})^{\varepsilon}w^{-1}$, where w is a word in x, y which can be assumed to be reduced. Now, let us assume that B is an invertible link. Then there is an automorphism ϕ of F_2 that carries x to a conjugate of its inverse, carries y to a conjugate of its inverse and carries $xyx^{-1}y^{-1}$ to its inverse (compare [29]). The abelianizing homomorphism λ maps F_2 onto the abelian group $Z \oplus Z$, and ϕ induces an automorphism ϕ' of $Z \oplus Z$. It is easy to see that ε is equal to the determinant of the matrix of ϕ' with respect to λx, λy. Therefore, it follows that $w(xyx^{-1}y^{-1})w^{-1} = yxy^{-1}x^{-1}$. But induction on the length of w shows that this is impossible. Thus B is a non-invertible link. The same argument implies that there is not an orientation-preserving involution of S^3 which induces an involution in each component of B and which keeps fixed exactly two points of B.

At this point, it may be useful to remark there is a possibility of existence of a knot N such that there is no orientation-preserving involution of S^3 which induces an involution on N. One of these possible knots seems to be 8_{17} (see [4] and [19]).

§3. *Generalized surgery on links*

In this section we will define modifications of the projection of a link L that generalize the modifications W_1, W_2 introduced earlier and also the ones defined in [10]. These modifications have the effect of exhibiting $\tilde{L}$ as a manifold which is obtained by doing generalized surgery on a link in S^3, that is, removing n disjoint solid tori from S^3 and replacing each torus with a special "graph-manifold" which is bounded by a torus. The advantage of this is that if a link has property P, then it will be shown that a counterexample to the Poincaré Conjecture cannot be obtained by doing generalized surgery on it (Theorem 4). This fact will allow us to establish Conjecture 1 for a large set of knots (see Section 4).

Let R be a finite tree with a distinguished vertex v(R) (the *origin of* R). The tree is to be valued as follows: each vertex of R is labeled either with a hyphen, or with an arbitrary integer, in such a way that each vertex labeled with a hyphen belong to exactly one edge, and the origin v(R) is always labeled with an integer. Each edge of R is labeled with a pair of coprime integers (α, β) where $0 \leq \beta \leq \alpha$. We call R a *valued tree*.

We will describe a procedure for assigning to each valued tree R a manifold W(R), such that $\partial W(R)$ is a torus with a fixed oriented fiber, and moreover such that W(R) is a 2-fold cyclic covering of a 3-ball B, which is branched over a system of curves L(R) such that $\partial L(R)$ is the set $\{a, b, c, d\}$ of Figure 8. To do this, we need some definitions.

Let $M(s, m)$ be a manifold obtained as follows. Let M be the S^1-bundle over S^2 which admits a section, and let H be a fiber of M. Suppose that S^2 and H have a fixed orientation. We remove $m + 2$ fibered solid tori V_i from $M, i = -1, 0, 1, \cdots, m$. Then, S^2 cuts ∂V_i

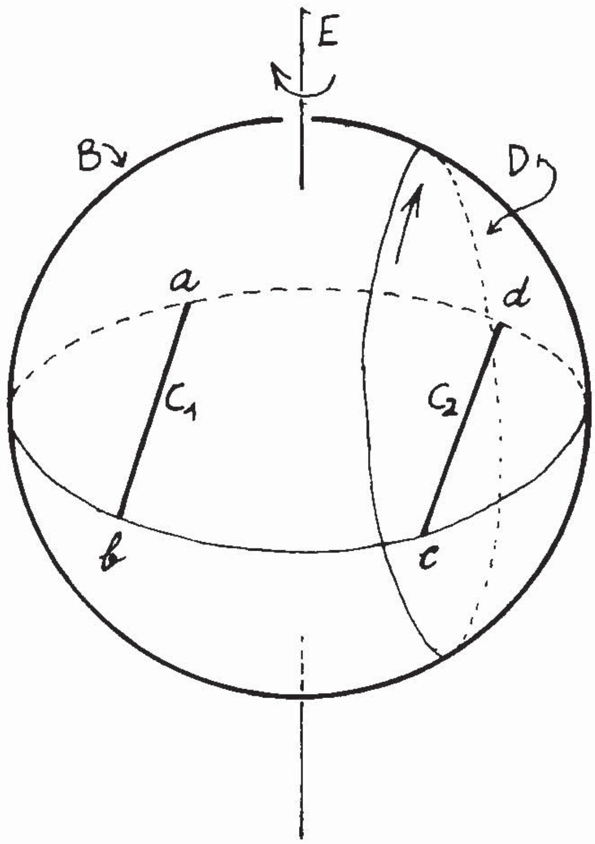

Fig. 8.

in a meridian curve m_i of V_i and we give to m_i the orientation induced by $S^2 - \text{int } V_i$. Let us take in ∂V_i a fiber h_i, with the orientation inherited from H. In order to obtain M(s, m) we now paste a solid torus in such a way that its meridian curve is homologous to $m_i + sh_i$, $i = -1$. The boundary of M(s, m) is $\bigcup_{i=0}^{m} \partial V_i$, and m_i, h_i are fixed oriented curves in ∂V_i. M(s, m) is a 2-fold cyclic covering of $B - \text{int}(B_1 \cup \cdots \cup B_m)$ branched over the curves L(s, m) of Figure 9 (for further details on the construction, see [18, Section 2 and Section 3]).

Let B be the ball of Figure 8. We define an autohomeomorphism t of ∂B as the composition of a rotation, of angle $\pi/2$ about the axis E which transforms a to d, and a symmetry with respect to the equatorial plane (see Figure 8). We define an autohomeomorphism v of ∂B as follows. Let D be a disc in ∂B which contains in its interior the points c, d and is disjoint from a, b (see Figure 8). Then, $v|D$ is

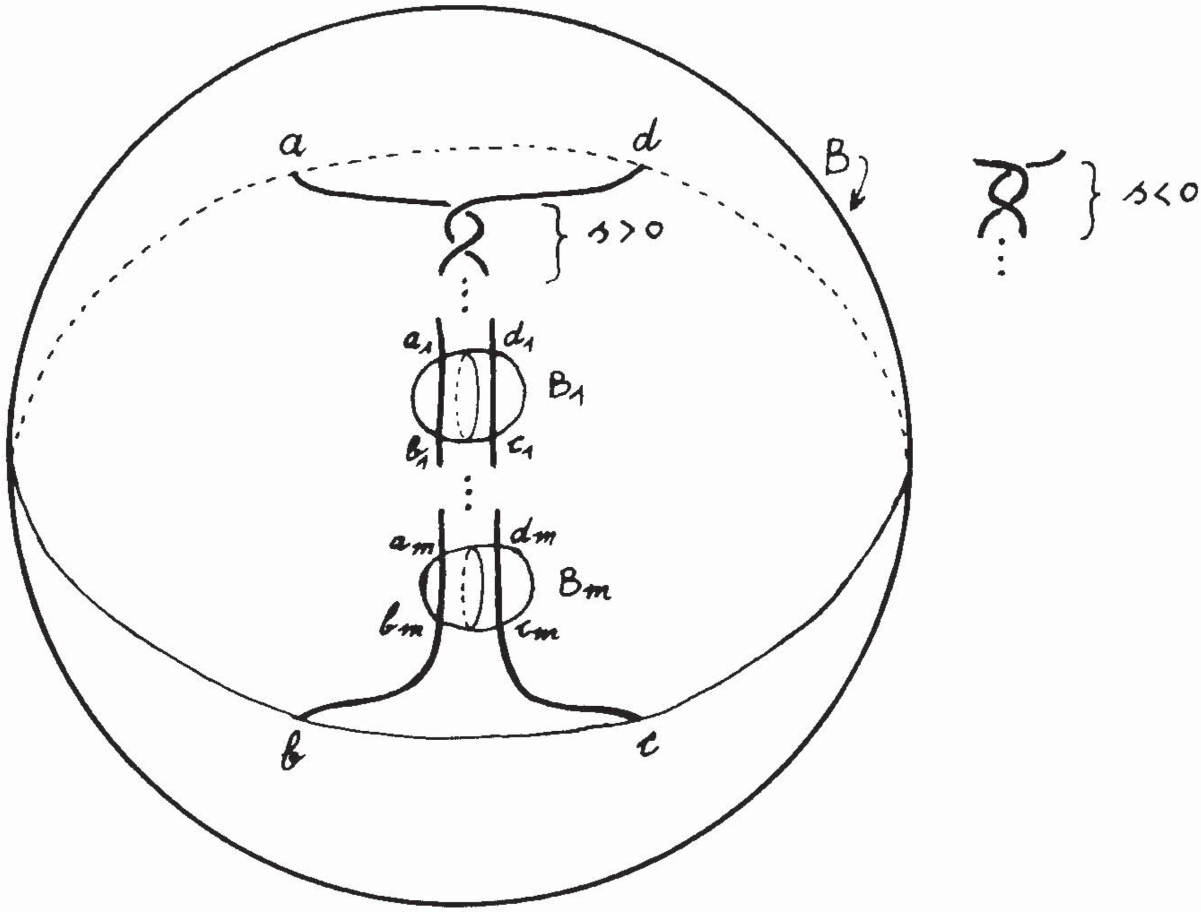

Fig. 9.

defined to be a "twist," holding D fixed, in the direction that is indicated in Figure 8, in order to move c to d. Now v is extended by the identity map outside D.

Now let α, β be two coprime integers. If α/β is the continued fraction

$$n + \frac{1}{m+} \cdots \frac{1}{+j}\ \frac{1}{+i}\ ,$$

we define an autohomeomorphism $g(\alpha,\beta)$ of ∂B as the composition $g(\alpha,\beta) = v^n t v^m t \cdots t v^j t v^i$, where v^0 is the identity map. Let $f(\alpha,\beta) = g(\alpha,\beta)t$. Extend the homeomorphisms t, v to B. Then, $g(\alpha,\beta)$ and $f(\alpha,\beta)$ admit an extension to B, which we denote with the same symbols $g(\alpha,\beta)$, $f(\alpha,\beta)$.

We are now ready to define W(R) and L(R) by induction on the number n of vertices of R which are labeled with an integer.

Let $v(R)$ be labeled with the integer s. Let us suppose also that $v(R)$ belongs to m edges $t_1,\cdots,t_m$ and that t_i is labeled with (α_i,β_i). Let u_i denote another vertex of t_i and assume that u_i, where $1 \le i \le r$, is labeled with a hyphen and that u_j, where $r+1 \le j \le m$, is labeled with an integer. Then, u_j, $r+1 \le j \le m$, is the "beginning" of a valued tree R_j.

Let $v(R_j) = u_j$. Note that the number of vertices of R_j which are labeled with an integer is $< n$. $W(R)$ is defined inductively, pasting the r solid torus $V_1,\cdots,V_r$ and the $m-r$ manifolds $W(R_j)$, $r+1 \le j \le m$, to $M(s,m)$ in such a way that a meridian curve of V_i is homologous to $\alpha_i m_i + \beta_i h_i$, and the oriented fiber, fixed in $\partial W(R_j)$, is homologous to $\alpha_j m_j + \beta_j h_j$. Note that in $\partial W(R) = \partial V_0$, the oriented fiber h_0 remains fixed. Then $L(R)$ is obtained replacing $f(\alpha_i,\beta_i)(L(s,0) \cap B_i)$, where $1 \le i \le r$, by $L(s,0) \cap B_i$ and replacing $g(\alpha_j,\beta_j)L(R_j)$, where $r+1 \le j \le m$, by $L(s,0) \cap B_j$ (see Figure 9). As an illustration of this process see the example of Figure 10.

Let L be a link in S^3 having m components $N_1,\cdots,N_m$. We will say that a 3-manifold M is obtained by doing *general surgery* m times on L if M is obtained by removing from S^3 a regular neighborhood $U(N_i)$ of N_i, $1 \le i \le m$, and replacing it with $W(R_i)$, where R_i is some valued tree, by pasting $\partial W(R_i)$ to $\partial(S^3 - U(N_i))$.

Let L be a link in S^3 and let us suppose that there is a ball B in S^3 such that $\partial(B \cap L)$ is the set $\{a,b,c,d\}$ (see Figure 8) and $B \cap L$ is a system of curves $g(\alpha,\beta)L(R)$, where R is an arbitrary valued tree and α,β are an arbitrary pair of coprime integers. We will say that has made a *general modification* on L, if we replace $B \cap L$ for the pair of curves C_1, C_2 of Figure 8. Let m be the minimum number of general modifications which have to be applied to L in order to change L into the trivial knot. It is clear that $\tilde{L}$ has been obtained by doing general surgery on a strongly-invertible link in S^3 of m components.

The following theorem is proved in the same way as Theorem 1:

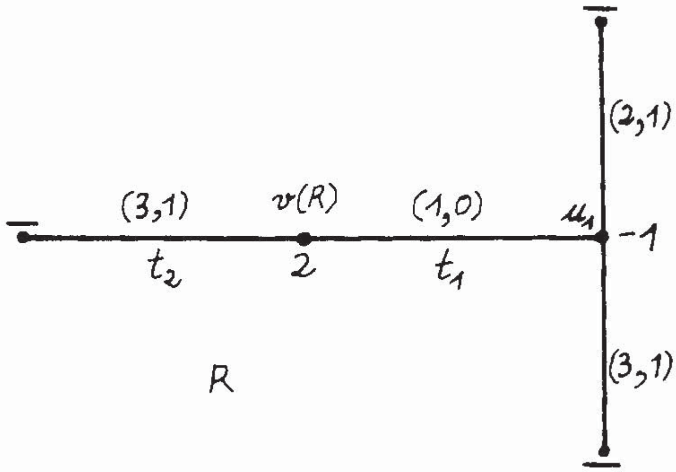

Fig. 10a.

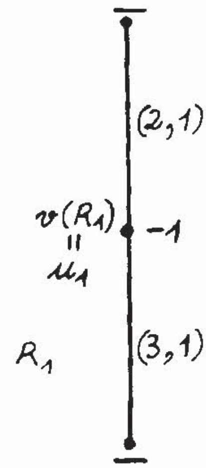

Fig. 10b.

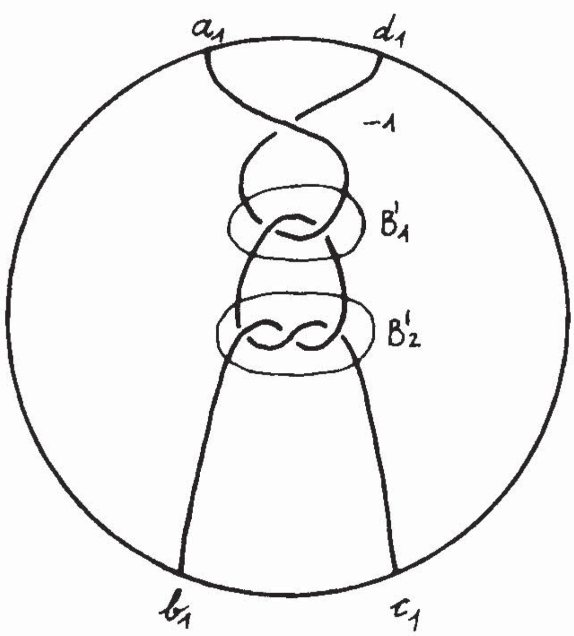

Fig. 10c.

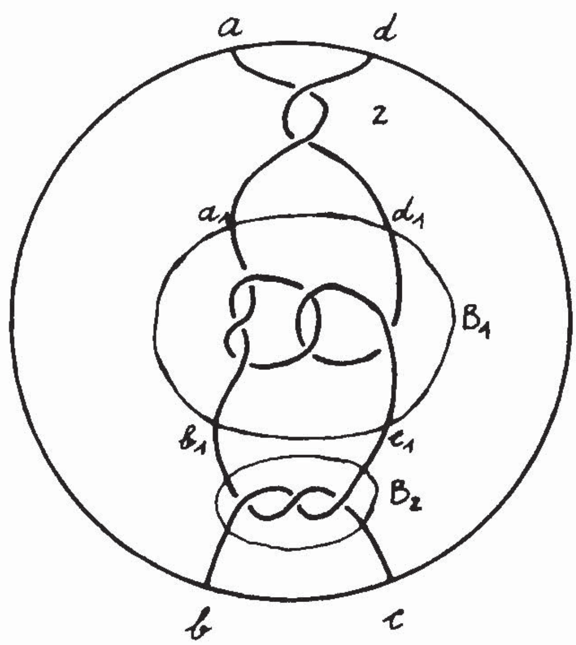

Fig. 10d.

THEOREM 3. *Every manifold that is obtained by doing general surgery on a strongly-invertible link is a 2-fold cyclic branched covering of* S^3.

The following theorem indicates a useful application of general surgery.

THEOREM 4. *If* M *is a simply-connected 3-manifold that is obtained by doing general surgery on a link* L *with property* P, *then* $M = S^3$.

In order to prove Theorem 4 we first need the following Lemma:

LEMMA 1. *Every homotopy 3-ball that lies in a graph-manifold is a 3-ball.*

We defer the proof of this lemma until Section 5.

Proof of Theorem 4. We are going to demonstrate the theorem by induction on the number n of graph-manifolds distinct from a solid torus which are introduced by surgery. If $n = 0$, there is nothing to prove, thus let $n > 0$. Let L_1 be a component of L such that a regular neighborhood, $U(L_1)$, of L_1 has been replaced by a graph-manifold $W(R)$ which is not a solid torus. If $\pi(M) = 1$, then $\partial U(L_1)$ bounds in M a homotopy solid torus ([1] and [8; Lemma 5.1]). If $W(R)$ were a homotopy solid torus, it would be a solid torus (by Lemma 1), hence $M - \text{int } W(R)$ is a homotopy solid torus. Then, $\pi(M - \text{int } W(R))$ is an infinite cyclic group with one generator which is represented by a simple curve C in $\partial(M - \text{int } W(R))$. We paste a solid torus to $M - \text{int } W(R)$ in such a way that C is a meridian curve of it. Thus we have built a manifold M', with $\pi(M') = 1$, which is obtained from S^3 by doing surgery on the link L, and replacing $n-1$ components of L by $n-1$ graph-manifolds which are not solid tori. By the induction hypothesis, $M' = S^3$ and thus $M - \text{int } W(R)$ is a solid torus. Therefore, M is a graph-manifold. Making use of the result of Lemma 1 we conclude that Theorem 4 is true. □

With the purpose of justifying the definitions of general modifications and general surgery, we make the following remarks. Let K be a non-trivial knot in S^3. If we wish to check Conjecture 1 for K, we can, for instance, apply m modifications of type W_2 in order to change K into the trivial knot. Then, $\tilde{K}$ is a manifold that is obtained by doing surgery on a strongly invertible link in S^3 of m components. By doing this in all possible ways, we obtain a family $\mathfrak{L}(K)$ of links in S^3 such that $\tilde{K}$ can be exhibited as a manifold obtained by doing surgery on an arbitrary member of $\mathfrak{L}(K)$. Let $m(K)$ be the minimal number of modifications of type W_2 which we have to apply to K in order to change K into the trivial knot. We define $\mathcal{L}'(K)$, $m'(K)$, in the same way as $\mathcal{L}(K)$ and $m(K)$, but replacing modifications of type W_2 for general modifications. Thus $\tilde{K}$ can be exhibited as a manifold obtained doing *general* surgery on an arbitrary member of $\mathcal{L}'(K)$.

As a consequence of Theorem 4, if a member of $\mathcal{L}'(K)$ has property P, then $\pi(\tilde{K}) \neq 1$. On the one hand $m'(K) \leq m(K)$ and this makes it easier to check Conjecture 1 for K in many cases, especially when $m'(K) = 1$, because property P has been intensively studied for knots. On the other hand, $\mathcal{L}(K) \subset \mathcal{L}'(K)$ and this increases our possibilities of finding a link with property P such that $\tilde{K}$ is obtained by doing general surgery on it.

It could happen that $m'(K) = 1$, for every non-trivial knot K. If this was so, then every 2-fold cyclic covering branched over a knot of S^3, would be obtained by doing general surgery on a strongly-invertible knot of S^3. Then, Conjecture 1 would be equivalent to the conjecture that every strongly-invertible knot has property P.

§4. *Applications*

If one seeks a counterexample to the Poincaré Conjecture among the 2-fold branched coverings of S^3, it is natural to examine covering spaces which are branched over knots which share deep properties with the trivial knot. One such property is that the trivial knot has Alexander polynomial $\Delta(t) = 1$. Note that if a knot N has Alexander polynomial $\Delta(t) = 1$ then $\tilde{N}$ is a homology 3-sphere.

1. *Kinoshita-Terasaka knots*

Let us consider the knots of Kinoshita-Terasaka [11, p. 149] $k(p,2n)$ ($k(3,6)$ is illustrated in Figure 11a or 11b). All of them have Alexander polynomial $\Delta(t) = 1$. Note that $k(3,6)$ can be obtained from the link of Figure 11c by substituting B_i for C_i $(i = 1,2,3)$. Thus [18] $\tilde{k}(3,6)$ is the graph-manifold that is represented (in Waldhausen's notation) by the graph of Figure 12, where $p = 3$, $n = 2$. In general, for $k(p,2n)$, $\tilde{k}(p,2n)$

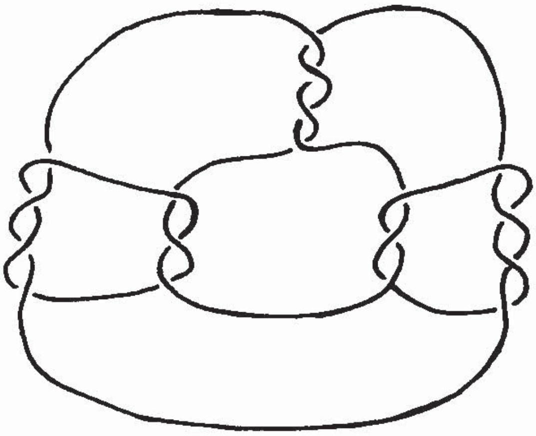

Fig. 11a.

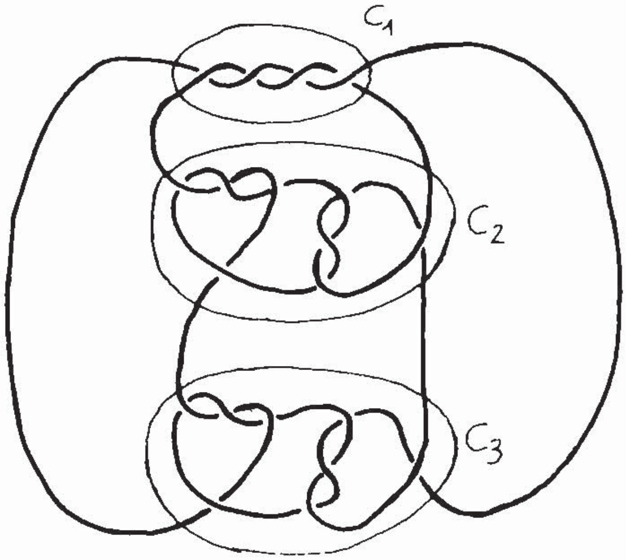

Fig. 11b.

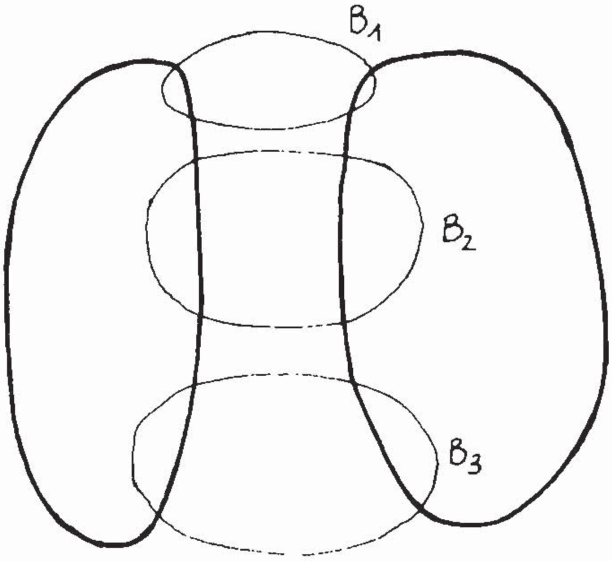

Fig. 11c.

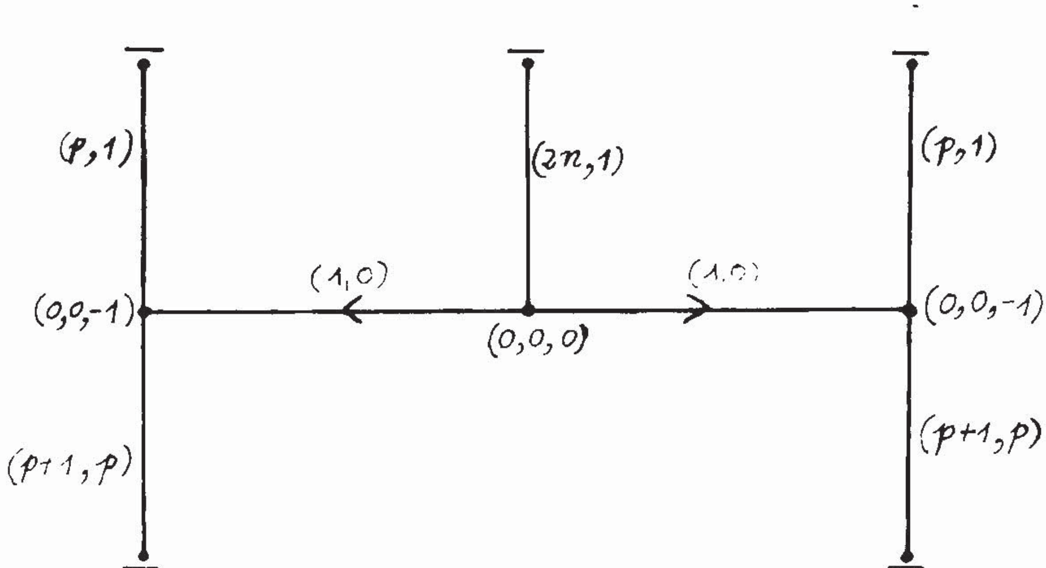

Fig. 12.

is the graph-manifold represented by the graph of Figure 12. Thus, by Lemma 1, $k(p, 2n)$ cannot give a counterexample to the Poincaré Conjecture.

2. *Conway's 11-crossing knot*

Let L be the knot, with Alexander polynomial $\Delta(t) = 1$, of Figure 13a, which was discovered by J. Conway in his enumeration of the non-alternating 11-crossing knots [3] (see also [20, p. 615]).

The trivial knot T can be obtained by doing one general modification in L (see Figure 13a, b). The 2-fold cyclic covering $\tilde{B}$ (resp $\tilde{C}$) of the ball B (resp. C) branched over $B \cap L$ (resp. $C \cap L$) is a solid torus. Then, $\tilde{L}$ can be obtained by removing $\tilde{C}$ from $\tilde{T} = S^3$ and sewing it back differently. The position of the ball C with respect to the trivial knot T is shown in Figure 14a. Then, $\tilde{C}$ is a regular neighborhood of the square knot (Figure 14b). Thus, $\tilde{L}$ can be obtained by doing surgery on the square knot, hence $\pi(\tilde{L}) \neq 1$, because a composite knot has property P ([1], [8]).

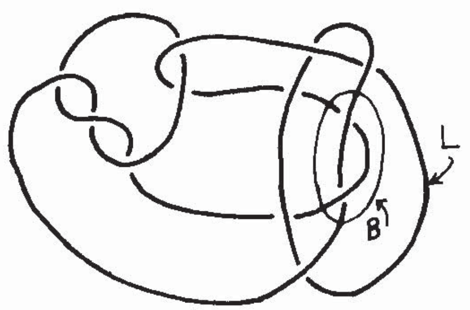

Fig. 13a.

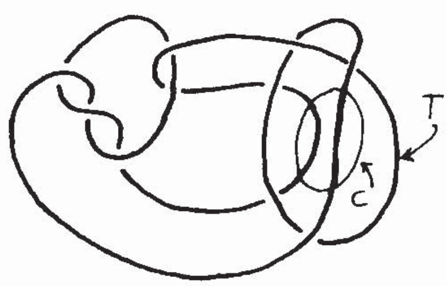

Fig. 13b.

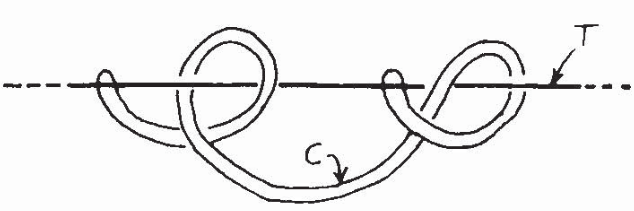

Fig. 14a.

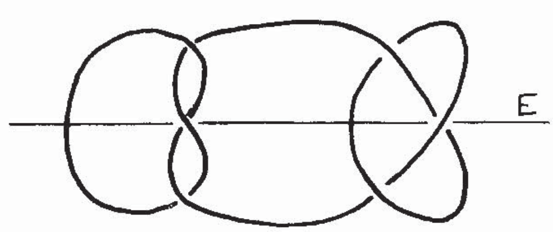

Fig. 14b.

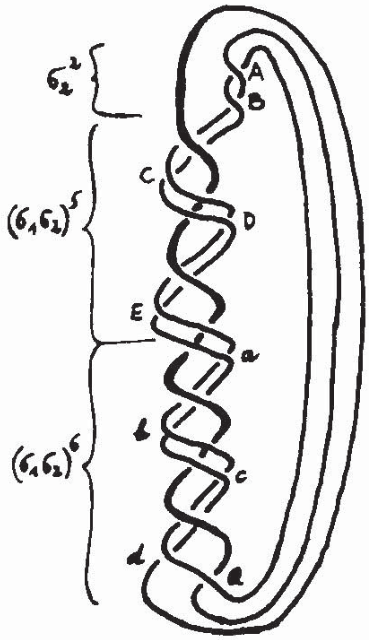

Fig. 15a.

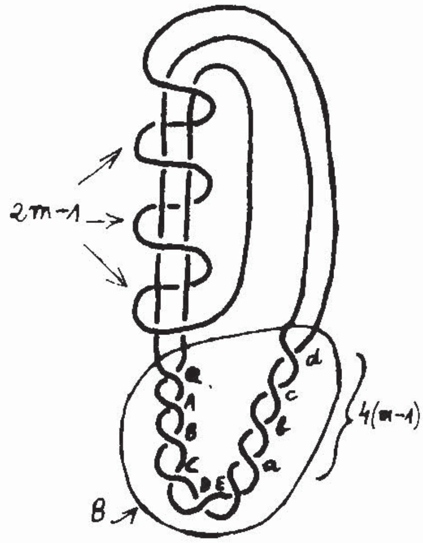

Fig. 15b.

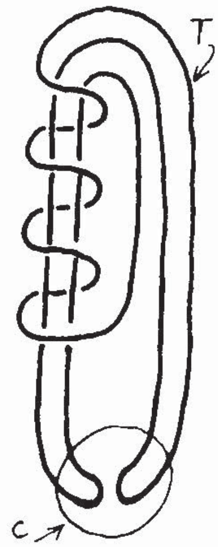

Fig. 15c.

3. *The 3-braid knots* $(\sigma_2\sigma_1^{-1})(\sigma_1\sigma_2)^{6m}$, $m \geq 1$

In [2] is is proved by Birman and Hilden that if Conjecture 1 is true for the knots $(\sigma_2\sigma_1^{-1})(\sigma_1\sigma_2)^{6m}$, $m \geq 1$ then Conjecture 1 is true for every 3-braid knot. We prove now that Conjecture 1 is true for the knots $(\sigma_2\sigma_1^{-1})(\sigma_1\sigma_2)^{6m}$, $m \geq 1$. For the sake of brevity, let L be the knot $(\sigma_2\sigma_1^{-1})(\sigma_1\sigma_2)^{12}$ of Figure 15a, b. The trivial knot T can be obtained by doing one general modification in L (see Figure 15b, c). The 2-fold cyclic covering $\tilde{B}$ (resp. $\tilde{C}$) of the ball B (resp. C) branched over $B \cap L$ (resp. $C \cap L$) is a solid torus. The position of the ball C with respect to the trivial knot T is shown in Figure 16a. Then, $\tilde{C}$ is a regular neighborhood of the twist knot T_3 (Figure 16b). Hence $\tilde{L}$ can be obtained by doing surgery on the twist knot T_3, hence $\pi(\tilde{L}) \neq 1$ because a twist knot has property P ([1], [8]).

A similar argument applies to the case where m is arbitrary. In general, the 2-fold cyclic covering branched over the 3-braid knot $(\sigma_2\sigma_1^{-1})(\sigma_1\sigma_2)^{6m}$ can be obtained by surgery on the twist knot T_{2m-1}.

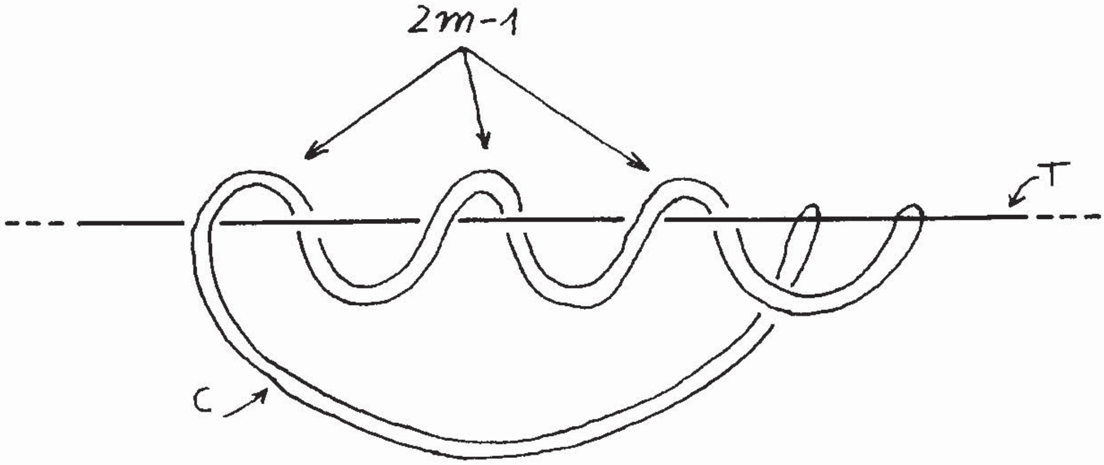

Fig. 16a.

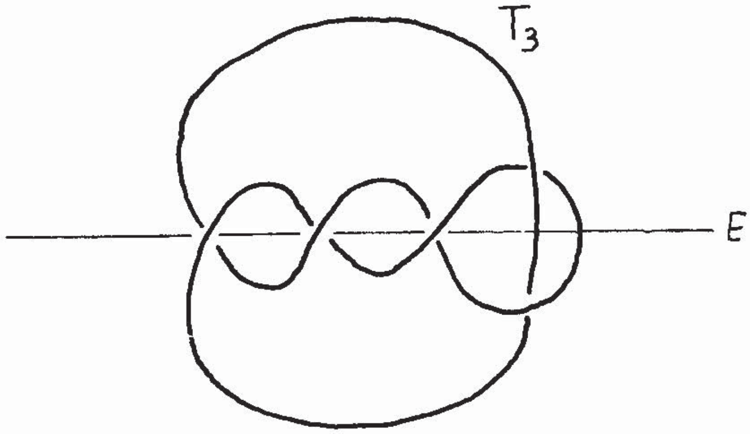

Fig. 16b.

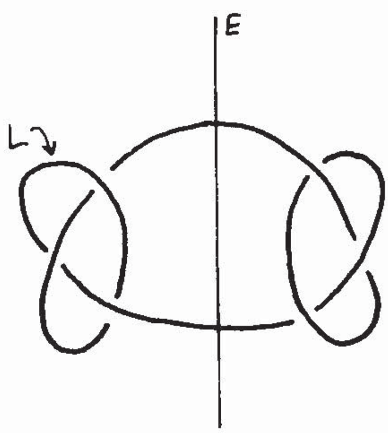

Fig. 17a.

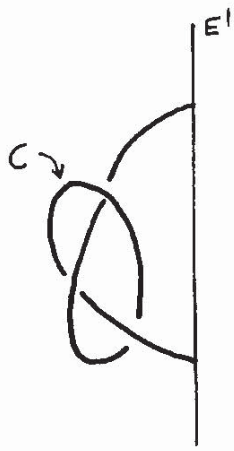

Fig. 17b.

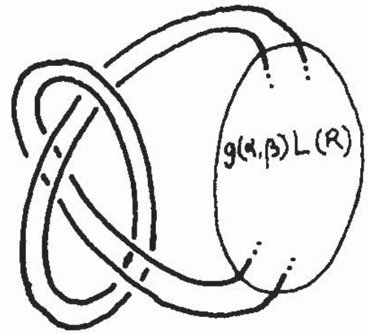

Fig. 17c.

4. *Generalized doubled knots*

Let L be the knot of Figure 17a. L is a strongly-invertible knot because the symmetry u with respect to the axis E leaves L invariant. Let $p: S^3 \to S^3$ be the 2-fold cyclic branched covering induced by u. Then, p(L) is the path C of Figure 17b. As a composite knot has property P ([1], [8]) then Conjecture 1 is true for the family of links of Figure 17c, where R is an arbitrary valued tree and where α, β are an arbitrary pair of coprime integers. As the same argument can be applied to an arbitrary strongly-invertible composite knot, we obtain in particular, that Conjecture 1 is true for every doubled knot (a fact proved by algebraic methods by Giffen [7]).

The same method can be applied to an arbitrary strongly-invertible link with property P (examples of these can be found in [1], [8] and [23]).

5. The idea illustrated in the following example may be useful. Let N be the knot of Figure 18 and let us consider a plane P with cuts N in the set $\{a, b, c, d\}$. Thus P divides S^3 into two balls A, B. The 2-fold cyclic covering $\tilde{A}$ (resp. $\tilde{B}$) of A (resp. B), branched over $A \cap N$ (resp. $B \cap N$) is the complement of a regular neighborhood of a non-trivial knot in S^3 (see Section 4.4.). Then, $\tilde{N}$ can be obtained by pasting $\partial\tilde{A}$ to $\partial\tilde{B}$. According to [1] and [8; Lemma 5.1] $\pi(\tilde{N}) \neq 1$.

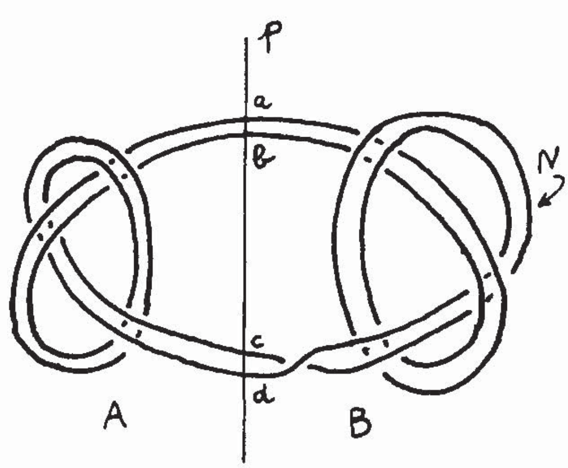

Fig. 18.

§5. *Demonstration of Lemma 1* *

Recall that the only simply connected Seifert manifold is S^3 [22].

On the one hand, every graph-manifold with boundary is a submanifold of a graph-manifold without boundary. On the other hand, every graph-manifold without boundary is [26, Satz 6.3] a connected sum of lens-spaces and *reduced graph-manifolds* ("Reduzierte Graphenmannigfaltigkeiten," see [26, 6.2]). Then, according to [9] and [26, Satz 7.1], Lemma 1 will be proved if we can show that a simply-connected, reduced, closed graph-manifold is S^3.

A reduced graph-manifold is either defined by a graph A(M) (see [26; 9]), or is a torus-bundle over S^1, or is a Seifert manifold over S^2 with three exceptional fibers. Thus, according to [9], it is sufficient to prove Lemma 1 for closed, reduced graph-manifolds M defined by a graph A(M). All of them [18; 7.5] are 2-fold cyclic coverings branched over a 3-sphere with g handles. If the graph A(M) is not a tree, or if any of the vertices of A(M) are valued with a triple $(g_j, 0, s_j)$, $g_j > 0$, then $g > 0$, hence $H_1(M) \neq 0$. If the graph A(M) is a tree with its vertices valued with triples $(g_j, 0, s_j)$, $g_j \leq 0$, then M is a 2-fold cyclic covering branched over a link L of S^3 [18; 7.3]. This link L has more than one component if $g_j < 0$ for any j [18; §3]. In this case, we have $H_1(M) \neq 0$. Then, let M be represented by a tree A(M) whose vertices are valued with triples $(0, 0, s_j)$. For [26, 9.2.3., 9.2.4.a), b) and c)] the vertices of A(M) either are of order ≥ 3, or are valued with a hyphen but there is always a vertex of order ≥ 3. We are going to prove Lemma 1, for those manifolds, by induction on the number m of vertices of order ≥ 3. If $m = 1$, M is a Seifert manifold and there is nothing to prove. Assume that $m > 1$. Then, there is a torus in M that splits M into two reduced graph-manifolds, M_1, M_2, corresponding to the graphs $A(M_1)$, $A(M_2)$ respectively. In order to build $A(M_1)$, $A(M_2)$ it is sufficient to

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* In this section we will follow the notation of Waldhausen in [26].

remove from $A(M)$ an edge which joins two vertices of order ≥ 3 and to value these vertices again with $(0,1,-)$. Then, $A(M_i)$, $i = 1,2$, has at least one vertex of order ≥ 2, valued with $(0,1,-)$.

According to [1], [8; Lemma 5.1], if $\pi(M) = 1$, then either M_1 or M_2 is a homotopy solid torus. We may assume that M_1 is a homotopy solid torus. Then, M_1 may be considered as a submanifold of either a Seifert manifold with three exceptional fibers, or a graph-manifold that is represented by a graph with $n < m$ vertices of order ≥ 3. Thus, by the induction hypothesis and according to [9; 2.2], M_1 is a solid torus. But then, [26; Satz, 9.4] $A(M_1)$ is a graph which has exactly one vertex of order zero, valued with $(0,1,-)$. This is a contradiction, hence $\pi(M) \neq 1$. □

Therefore, a simply connected graph-manifold is S^3.

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